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About Weakly Uniformly Paracompact Spaces

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Abstract. In this work we introduce and study weakly uniformly paracompact spaces. In particular, the characterizations of weakly uniformly paracompact spaces by using Hausdorff compact extensions and ω-mapping are obtained.
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INTRODUCTION

Throughout this work all uniform spaces are assumed to be Hausdorff, topological spaces Tychonoff and mappings are uniformly continuous.

For coverings α and β of a set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e. for any $A \in \alpha$ there exist $B \in \beta$ such that $A \subset B$. The covering α is called finitely additive, if $\alpha^{\angle} = \alpha$, where $\alpha^{\angle} = \{\bigcup \alpha_0 : \alpha_0 \subset \alpha \text{ is finite}\}$.

A uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called:

(1) precompact, if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and a finite uniform covering $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [4];

(2) uniformly perfect if it is both precompact and perfect [4];

Let ω be an open covering of a topological space X to the topological space Y. A mapping f is called an ω-mapping if every point $y \in Y$ has a neighborhood O_y whose inverse image $f^{-1}O_y$ is contained in at least one element of the covering ω [1].

A covering α of a topological space X is called point-finite if every point of X lies in only finitely many members of α [3]. A uniform space (X, U) is called uniformly A-paracompact if every its finitely additive open covering has a locally finite uniform refinement [2].

For the uniformity U, by τ_U we denote the topology generated by the uniformity and symbol U_X means the universal uniformity.

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Let (X, U) be a uniform space.

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Definition 1 A uniform space (X, U) is called weakly uniformly paracompact if every finitely additive open covering of X has a point-finite uniform refinement.

Proposition 1 *If (X, U) is a weakly uniformly paracompact space, then the topological space (X, τ_U) is weakly paracompact. Conversely, if (X, τ) is weakly paracompact, then the uniform space (X, U_X), where U_X is the universal uniformity, is weakly uniformly paracompact.*

Proof. Let α be an arbitrary open covering of the space (X, τ_U). Then, for a finitely additive open covering $\alpha^{\angle}$ of the uniform space (X, U) there exists a point-finite uniform covering $\beta \in U$ which is a refinement of it. It is known that the interior $\langle\beta\rangle = \{\langle B\rangle : B \in \beta\}$ of the uniform covering β is a uniform covering, where $\langle B\rangle$ is the interior of the set B. Let $\gamma = \langle\beta\rangle$. It is clear that γ is a point-finite open uniform covering of (X, U). For each $\Gamma \in \gamma$ choose $A_\Gamma \in \alpha_{\aleph_0}$ such that $\Gamma \subset A_\Gamma$, where $A_\Gamma = \bigcup_{i=1}^{n} A_i$, $A_i \in \alpha$, $i = 1, 2, ..., n$. Let $\alpha_0 = \bigcup\{\alpha_\Gamma : \Gamma \in \gamma\}$, $\alpha_\Gamma = \{\Gamma \bigcap A_i : i = 1, 2, ..., n\}$. Then α_0 is a point-finite open covering of the space (X, τ_U), and it is a refinement of α. So, the space (X, τ_U) is weakly paracompact.

Conversely, let the Tychonoff space (X, τ) be weakly paracompact. Then the set of all open coverings forms the base of the universal uniformity U_X of the space (X, τ). It is easy to see that the uniform space (X, U_X) is weakly uniformly paracompact.

The Japanese mathematician G. Tamano gave a remarkable characterization of paracompact spaces in terms of compact extensions.

The following theorem gives a characterization of weak uniform paracompactness in the spirit of Tamano.

Theorem 1 *Let (X, U) be a uniform space and bX be a certain its compact Hausdorff extension. The uniform space (X, U) is weakly uniformly paracompact if and only if for each compactum $K \subset bX \setminus X$ there exists a point-finite uniform covering $\alpha \in U$ such that $[A]_{bX} \bigcap K = \varnothing$ for all $A \in \alpha$.*

Proof. *Necessity*. Let (X, U) be weakly uniformly paracompact and $K \subset bX \setminus X$ be an arbitrary compactum. Then for each point $x \in X$ there is an open neighborhood O_x in bX such that $[O_x]_{bX} \bigcap K = \emptyset$. It is clear that $\gamma = \{O_x \bigcap X : x \in X\}$ is an open covering of the uniform space (X, U). We form an open covering $\gamma^{\angle}$ of the (X, U), taking as elements of γ. Then $\gamma^{\angle}$ is a finitely additive open covering of the space (X, U). According to the condition of the theorem, it is possible to refine a covering $\gamma^{\angle}$ by a point-finite uniform covering $\beta \in U$. Then $[B]_{bX} \subset [\bigcup_{i=1}^{n}(O_{x_i} \bigcap X)]_{bX} \subset \bigcup_{i=1}^{n}[O_{x_i}]_{bX}$. As $[O_{x_i}]_{bX} \bigcap K = \emptyset$ for any $i = 1, 2, ..., n$, then $[B]_{bX} \bigcap K = \varnothing$ for any $B \in \beta$.

Sufficiency. Let α be an arbitrary finitely additive open covering of a space (X, U). Then there is an open family β in bX such that $\beta \wedge \{X\} = \alpha$. Let $K = bX \setminus \bigcup\beta$. It follows that K is compactum. Then, by the condition of the theorem, there exists a point-finite uniform covering $\gamma \in U$ such that $[\Gamma]_{bX} \bigcap K = \emptyset$ for any $\Gamma \in \gamma$. Since $[\Gamma]_{bX}$ is compactum in bX there are $B_1, B_2, ..., B_n \in \beta$ such that $[\Gamma]_{bX} \subset \bigcup_{i=1}^{n} B_i$. Then $\Gamma \subset \bigcup_{i=1}^{n} A_i$, where $\bigcup_{i=1}^{n} A_i \in \alpha$. Consequently, (X, U) is a weakly uniformly paracompact space.

Definition 2 A uniform space (X, U) is called uniformly B-locally compact, if there exists a point-finite uniform covering consisting of compact subsets.

The next theorem gives a connection between weak uniform paracompactness and uniform B-locally compactness.

Theorem 2 *Any uniformly B-locally compact space is weakly uniformly paracompact.*

Proof. Let α be an arbitrary finitely additive open covering of the space (X, U). Then there exists a point-finite uniform covering β consisting of compact subsets. It is easy to see that the covering β is a refinement of α. Consequently, the space (X, U) is weakly uniformly paracompact.

The next two propositions show that weak uniform paracompactness is preserved when passing to a closed subspaces and any disjoint sum of uniform spaces.

Proposition 2 *Any closed subspace M of a weakly uniformly paracompact space (X, U) is weakly uniformly paracompact.*

Proof. Let γ be a finitely additive open covering of M. Let $\hat{\gamma}$ denote the open covering of the space (X, U), consists of all elements of the covering γ and the set $X \setminus M$. It is clear that $\hat{\gamma}$ is a finitely additive covering. According to the condition there exists a point-finite uniform covering $\beta \in U$ which is a refinement of $\hat{\gamma}$. Let β_M be the trace of β on M. It is easy to see that β_M is a uniform covering of the subspace M and is a refinement of γ. β_M is a point-finite covering. Indeed, let $x \in M$ be an arbitrary point. Since β is a point-finite uniform covering, then $x \in M \subset X$ belongs to only finitely many elements of the covering β. Then $x \in M$ belongs to only finitely many elements of the covering β_M. Thus, in any finitely additive open covering γ of the subspace M, it was possible to inscribe a point-finite uniform covering of β_M. Therefore, the subspace M is weakly uniformly paracompact.

Proposition 3 *The sum of any family of weakly uniformly paracompact spaces is weakly uniformly paracompact.*

Proof. Let $\{(X_a, U_a) : a \in M\}$ be an arbitrary family of weakly uniformly paracompact spaces (X_a, U_a) and $(\coprod_{a \in M} X_a, \coprod_{a \in M} U_a)$ be the sum of uniform spaces. Consider an arbitrary finitely additive open covering α of the space $(\coprod_{a \in M} X_a, \coprod_{a \in M} U_a)$. It is easy to see that the family $\beta = \{X_a \bigcap A : a \in M, A \in \alpha\}$ is again a finitely additive open covering of the space $(\coprod_{a \in M} X_a, \coprod_{a \in M} U_a)$ and is a refinement of α. For each $a_0 \in M$, put $\beta_{a_0} = \{X_{a_0} \bigcap A : a_0 \in M, A \in \alpha\}$. It is clear that it is a finitely additive open covering of the space (X_{a_0}, U_{a_0}), and therefore, there exists a point-finite uniform covering $\gamma_{a_0} \in U_{a_0}$ which is a refinement of β_{a_0}. Next, consider the family γ which is the union of all families γ_a, $a \in M$. Then the family γ is a uniform covering of the space $(\coprod_{a \in M} X_a, \coprod_{a \in M} U_a)$ and it is a refinement of α. We show that γ is point-finite. Let $x \in X$ be an arbitrary point. Let $x \in X_a$, $a \in M$. Since γ_a, $a \in M$ is a point-finite uniform covering of the space (X_a, U_a), $a \in M$, then $x \in X_a$ belongs to only finitely many elements of γ_a, $a \in M$. Since the spaces (X_a, U_a), $a \in M$, are disjoint, each point $x \in X$ belongs to at most finitel many elements of the covering γ.

The following theorem shows that strong uniform paracompactness is preserved in the preimage direction by uniformly perfect mappings.

Theorem 3 *Weak uniform paracompactness is preserved in the preimage direction by uniformly perfect mappings.*

Proof. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping from a uniform space (X, U) to a weakly uniformly paracompact space (Y, V). Let α be an arbitrary finitely additive open covering of (X, U). It is clear that the covering $\{f^{-1}y : y \in Y\}$ refines the covering α. Then $\beta = f^{\#}\alpha = \{f^{\#}A : A \in \alpha\}$, where $f^{\#}A = Y \setminus f(X \setminus A)$, is an open covering of the space (Y, V). Considering all possible finite unions of sets of β, we construct an open covering $\beta^{\angle}$. It is a finitely additive open covering. By the condition of the theorem, there is a point-finite uniform covering $\gamma \in V$ of it. It is easy to see that the covering $f^{-1}\beta^{\angle}$ is a refinement of the covering α. The $f^{-1}\gamma$ is a point-finite uniform covering of the space (X, U), and it is a refinement of α. So, the uniform space (X, U) is weakly uniformly paracompact.

The following theorem is a uniform analogue of Dowker-Ponomarev-Fedorcuk-Shediva's (Trnkova) theorem for weakly uniformly paracompact spaces.

Theorem 4 *A uniform space (X, U) is weakly uniformly paracompact if and only if for every finitely additive open covering ω of (X, U) there exists a uniformly continuous ω-mapping $f : (X, U) \to (Y, V)$ of (X, U) onto a metrizable weakly uniformly paracompact space (Y, V).*

Proof. *Necessity* Let (X, U) be a metrizable weakly uniformly paracompact space and ω be an arbitrary finitely additive open covering of X. Then the identity map of the space (X, U) is the required uniformly continuous ω-mapping of (X, U) onto a metrizable weakly uniformly paracompact space.

Sufficiency Let ω be an arbitrary finitely additive open covering of the space (X, U). Then there exists a uniformly ω-continuous mapping $f : (X, U) \to (Y, V)$ of (X, U) onto some metrizable weakly uniformly paracompact space (Y, V). For each point $y \in Y$, there exists a neighborhood O_y whose preimage $f^{-1}O_y$ is contained in some element of the covering ω. Let $\beta = \{O_y : y \in Y\}$. We form an open covering $\beta^{\angle}$ consisting of all possible finite unions of elements of β. Let $\gamma \in V$ be a point-finite uniform covering that refines $\beta^{\angle}$. Then the covering $f^{-1}\gamma$ is a refinement of the covering ω of (X, U) . We show that $f^{-1}(\gamma)$ is a point-finite uniformly covering. Indeed, let $x \in X$ be an arbitrary point in X and $y = f(x)$. Then the point $y \in Y$ belongs to a finite number of elements of the covering γ. It is easy to see that the point $x \in f^{-1}y$ belongs to only finitely many elements of the covering $f^{-1}\gamma$. Therefore, the uniform space (X, U) is weakly uniformly paracompact.

Theorem 5 *Any uniformly perfect mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) onto a uniform space (Y, V) is an ω-mapping for any finitely additive open covering ω of (X, U).*

Proof. Let ω be an arbitrary finitely additive open covering of the space (X, U). It is easy to see that the covering $\alpha = \{f^{-1}y : y \in Y\}$ is a refinement of ω. For each $f^{-1}y \in \alpha$, choose a $W_y \in \omega$ such that $f^{-1}y \subset W_y$. Then from the closedness of the mapping f there exists a neighborhood $O_y \setminus \{y\}$ such that $f^{-1}O_y \subset W_y$.

Proposition 4 *The product of a weakly uniformly paracompact uniform space (X, U) and a compact uniform space (Y, V) is weakly uniformly paracompact.*

Proof. Let (X, U) be a weakly uniformly paracompact space and (Y, V) be a compact uniform space. It is known [see 4, p. 77, Example 1.7.2] that the projection $\pi_X : (X, U) \times (Y, V) \to (X, U)$ is uniformly perfect. Then it is an ω-mapping of the product $(X, U) \times (Y, V)$ onto a weakly uniformly paracompact space (Y, V) for any finitely additive open covering ω of $(X, U) \times (Y, V)$. Therefore, according to Theorem 4, the uniform space $(X, U) \times (Y, V)$ is weakly uniformly paracompact.

Any uniformly A-paracompact space is weakly uniformly paracompact. The converse need not be true.

The following theorem is an intrinsic characterization of strongly uniformly paracompact spaces.

Theorem 6 *For a uniform space (X, U) the following are equivalent:*

1. *(X, U) is uniformly A-paracompact;*
2. *(X, U) is weakly uniformly paracompact and the topological space (X, τ_U) is paracompact.*

Proof. 1) $\Rightarrow$ 2) It is obviously.

2) $\Rightarrow$ 1). Let α be an arbitrary finitely additive open covering of the uniform space (X, U). There is a locally finite open covering β which is a refinement of α. We form the covering $\beta^{\angle}$ consisting of all possible finite unions of elements of β. Then $\beta^{\angle}$ is a finitely additive open locally finite covering. Next, there is a point-finite refinement γinU of $\beta^{\angle}$. Therefore, a locally finite uniform covering $\beta^{\angle}$ is a refinement of the finite additive open covering α. Thus, the uniform space (X, U) is uniformly A-paracompact.

REFERENCES

[1] P.S. Aleksandrov, *Introduction to the Theory of Sets and General Topology* (Nauka, Moscow, 1977; in Russian).

[2] L.V. Aparina, Uniformly Lindelöf Space, Trudy Mosk. Math. Obshestva **57**, 3–15 (1996) (in Russian).

[3] A.V. Arkhangel'sky, V.I. Ponomarev,*Foundation of General Topology in Problems and Exercises* (Nauka, Moscow, 1974; in Russian).

[4] A.A. Borubaev, P.S. Pankov, and A.A. Chekeev, *Spaces Uniformed by Coverings* (Budapest, 2003).