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Cite as: AIP Conference Proceedings **2325**, 020055 (2021); https://doi.org/10.1063/5.0040273
Published Online: 09 February 2021

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On Uniformly Lindelöf Spaces

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Abstract. As we know the Lindelöfness play an important role in the General Topology. Therefore, the finding of uniform analogues of Lindelöfness is an important and interesting problem in the theory of Uniform Topology. For example, uniform *B*-Lindelöfness in the sense of A. A. Borubaev [1], uniform *A*-Lindelöfness in the sense of L.V. Aparina [2], uniform *I*- Lindelöfness in the sense of D. R. Isbell [3].
In this article we show a new approach to the definition of a uniform analog of Lindelöfness. We introduce and study uniform *R*-Lindelöf and uniform Lindelöf spaces.

INTRODUCTION

Throughout this work all uniform spaces are assumed to be Hausdorff and mappings are uniformly continuous.

For coverings α and β of the set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, for coverings α and β of a set X, we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$. The covering α finitely additive if $\alpha^{\angle} = \alpha$, $\alpha^{\angle} = \{\bigcup \alpha_0 : \alpha_0 \subset \alpha \text{ is finite}\}$. $\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(H) = \bigcup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha : A \bigcap H \neq \varnothing\}$, $H \subset X$.

A uniform space (X,U) is called $\aleph_0$-bounded if the uniformity U has a base consisting of countable coverings; a uniform space (X,U) is called τ-bounded if the uniformity U has a base consisting of coverings of cardinality $\leq \tau$ [1], if $\tau \leq \aleph_0$, then the uniform space (X,U) is called uniformly *I*-Lindelöf [3]; a uniform space (X,U) is called uniformly *B*-Lindelöf, if it is both uniformly *B*-paracompact and $\aleph_0$-bounded [1]; a uniform space (X,U) is called uniformly *A*-Lindelöf, if for each open covering α exist a countable uniformly covering $\beta = \{B_n : n \in N\}$ and $\gamma \in U$ such that $\beta \succ \alpha^{\angle}$ and $\gamma(\bar{B}_n) \subset B_{n+1}$ for any $n \in N$ [2]; a uniformly continuous mapping $f : (X,U) \to (Y,V)$ of uniform space (X,U) onto a uniform space (Y,V) is called a precompact, if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and finite uniform covering $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$[1]; a uniformly continuous mapping $f : (X,U) \to (Y,V)$ of uniform space (X,U) onto a uniform space (Y,V) is called a uniformly perfect, if it is both precompact and perfect [1]; a uniformly continuous mapping $f : (X,U) \to (Y,V)$ of uniform space (X,U) onto a uniform space (Y,V) is called a uniformly open, if f maps each open uniform covering $\alpha \in U$ to an open uniform covering $f\alpha \in V$ [1]; a continuous mapping $f : X \to Y$ of topological space to a topological space called ω-mapping, if for each point $y \in Y$ there exist such neighborhood O_y and $W \in \omega$, that $f^{-1}O_y \subset W$ [1]; a uniform space (X,U) called uniformly *R*-paracompact, if every open covering has an open uniformly locally finite refinement [4]; a uniform space (X,U) called uniformly *B*-paracompact, if for each finitely additive open covering γ of (X,U) there exists such sequence uniform covering $\{\alpha_i : i \in N\} \subset U$, that following condition is realized: for each point $x \in X$ there exist such number $i \in N$ and $\Gamma \in \gamma$ that $\alpha_i(x) \subset \Gamma$ (*) [1]; a uniform space (X,U) called uniformly *P*-paracompact, if for each open cover γ of (X,U) there exists such sequence uniform covering $\{\alpha_i : i \in N\} \subset U$, that the condition (*) is realized [5]; a uniform space (X,U) is called uniformly *P*-Lindelöf, if for each open cover γ of (X,U) there exists such sequence countable uniform covering $\{\alpha_i : i \in N\} \subset U$, that the condition (*) is realized [6]; a uniform space (X,U) is called uniformly locally compact, if there exists a uniform covering $\alpha \in U$ consisting of compact subsets; a uniform space (X,U) is called uniformly locally Lindelöf [1], if there exists such uniform covering $\alpha \in U$ that the closures of all its elements are Lindelöf [5]; a uniform space (X,U) is called strongly uniformly *R*-paracompact if every open covering has an open uniformly star finite refinement. For the uniformity U by τ_U we denote the topology generated by the uniformity.

UNIFORMLY *R*-LINDELÖF SPACES AND THEIR GENERALIZATIONS

Let (X,U) be a uniform space.

Definition 1 *A uniform space (X,U) is said to be uniformly R-Lindelöf, if it is both uniformly R-paracompact and $\aleph_0$-bounded.*

International Conference on Analysis and Applied Mathematics (ICAAM 2020)
AIP Conf. Proc. 2325, 020055-1–020055-5; https://doi.org/10.1063/5.0040273
Published by AIP Publishing. 978-0-7354-4069-2/$30.00

Proposition 1 *If (X,U) is a uniformly R-Lindelöf space then the topological space (X,τ_U) is uniformly R-Lindelöf. Conversely, if (X,τ) is Lindelöf then the uniform space (X,U_X) is uniformly R-Lindelöf.*

Proof. Let α be an arbitrary open cover of the space (X,τ_U) . Then for the open covering α exists a uniformly locally finite open covering β which is a refinement of it. Since the space (X,U) is $\aleph_0$-bounded, there exists a countable cover λ such that for any $i \in N$ we have that $L_i \subset \bigcup_{j=1}^{k} A_j$. Then system $\{L_i \bigcap A_j\}$, $i=1,2,...,n,\ j=1,2,...,k$ forms a countable open covering which is refinement in covering α. Consequently, the space (X,τ_U) is Lindelöf.

Conversely, if (X,τ) is Lindelöf, then the system of all open coverings forms a base of universal uniformity U_X of the space (X,τ). It follows from this that (X,U_X) is uniformly R-Lindelöf.

Proposition 2 *Any compact space is uniformly R-Lindelöf.*

Proof. Since any compact uniform space is precompact, it is all the more $\aleph_0$-bounded. It is clear that every compact space is uniform R-paracompact. Concequently, the uniform space (X,U) is uniformly R-Lindelöf.

Proposition 3 *Each closed subspace of a uniformly R-Lindelöf space is uniformly R-Lindelöf.*

Proposition 4 *Any uniformly R-Lindelöf space is uniformly locally Lindelöf.*

Proposition 5 *Any uniformly locally compact and $\aleph_0$-bounded space is uniformly R-Lindelöf.*

Proposition 6 *Any uniformly R-Lindelöf space is uniformly B-Lindelöf.*

Proof. Let (X,U) be a uniformly R-Lindelöf space. As is known, every uniformly R-paracompact space is uniformly B-paracompact [1]. Hence, the uniform space is uniformly B-Lindelöf.

Theorem 1 *Any uniformly R-Lindelöf space is strongly uniformly R-paracompact.*

Proof. Let (X,U) be a uniformly R-Lindelöf space. Then from the fact that a space (X,U) is strongly uniformly R-paracompact if and only if it is uniformly R-paracompact and the space (X,τ_U) is strongly paracompact it follows that is strongly uniformly R-paracompact.

Proposition 7 *Any uniformly R-Lindelöf space is complete.*

Theorem 2 *Let $f:(X,U)\to(Y,V)$ be a uniformly perfect mapping of a uniform space (X,U) onto a uniform space (Y,V). Then uniformly R-Lindelöf space converse both to direction of image and to one of preimage.*

Proof. If a space (X,U) is $\aleph_0$-bounded, then its uniformly continuous image (Y,V) is also $\aleph_0$-bounded. Let a uniform space (X,U) is uniformly R-paracompact. Then by Theorem 2.3.9 [1, p. 155] we have the space (Y,V) is uniformly R-paracompact. Thus the uniform space (Y,V) is uniformly R-Lindelöf. Conversely, let the space (Y,V) is $\aleph_0$-bounded. Let $\alpha \in U$ be an arbitrary uniformly covering. Then by virtue of the perfectness of the mappings f there exist such a countable covering $\beta \in V$ and finite covering $\gamma \in U$ that $f^{-1}\beta \wedge \gamma \succ f^{-1}\alpha$. But the covering $f^{-1}\beta \wedge \gamma$ is countable. Therefore the space (X,U) is $\aleph_0$-bounded. The uniformly R-paracompactness of the space (X,U) follows from Theorem 2.3.9. [1, p. 155] Thus, (X,U) is uniformly R-Lindelöf.

Theorem 3 *Let $f:(X,U)\to(Y,V)$ be a uniformly open mapping of a uniform space (X,U) onto a uniform space (Y,V). If (X,U) is uniformly R-Lindelöf space then the uniform space (Y,V) is also uniformly R-Lindelöf.*

Proof. Let f be a uniformly open mapping of a uniform space (X,U) onto a uniform space (Y,V) and α be an arbitrary finite additive open covering. Then $f^{-1}\alpha$ is a finite additive open covering of the space (X,U) and by the criterion of uniformly R-paracompactness we have $f^{-1}\alpha \in U$. By virtue of the uniformly openness of the mapping f we have $\alpha \in V$. Hence, (Y,V) is uniformly R-paracompact. If a space (X,U) is $\aleph_0$-bounded, then (Y,V) is also $\aleph_0$-bounded. Consequently, the space (Y,V) is uniformly R-Lindelöf.

Definition 2 *A uniformly continuous mapping $f:(X,U)\to(Y,V)$ of a uniform space (X,U) onto a uniform space (Y,V) is said to be uniformly R-Lindelöf, if the following conditions is realized:*

1. *For each open covering α of the space (X,U) there exist such open covering β of the space (Y,V) and uniformly locally finite open covering γ of the space (X,U), that the covering $f^{-1}\beta \wedge \gamma$ is refined in a covering α;*

2. *For each $\lambda \in U$ there exist such $\eta \in V$ and countable covering $\mu \in U$, that the covering $f^{-1}\eta \wedge \mu$ is refined in a covering λ.*

Proposition 8 *Let $f : (X,U) \to (Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) onto a uniform space (Y,V). If (X,U) is uniformly R-Lindelöf space then the uniformly continuous mapping f is uniformly R-Lindelöf.*

Proof. Let (X,U) be a uniformly R-Lindelöf space and α be an arbitrary open covering. Then exist such uniformly locally finite open covering γ of the space (X,U), that the covering γ is refined in a covering α. For open covering β of the space (Y,V) we have the covering $f^{-1}\beta \wedge \gamma$ is refined in a covering α. Let $\lambda \in U$ be an arbitrary uniform covering. By virtue of the $\aleph_0$-boundedness of the uniform space (X,U) exist such countable uniform covering $\mu \in U$, that the covering μ is refined in a covering λ. Then for uniform covering $\eta \in V$ we have that the uniform covering $f^{-1}\eta \wedge \mu$ is refined in a uniform covering λ. Consequently, the mapping f is uniformly R-Lindelöf.

Proposition 9 *If uniformly continuous mapping $f : (X,U) \to (Y,V)$ of a uniform space (X,U) onto a uniform space (Y,V) , $Y = \{y\}$ is uniformly R-Lindelöf, then the uniform space (X,U) is uniformly R-Lindelöf.*

Proof. Let f be a uniformly R-Lindelöf mapping and α be an arbitrary open covering of the space (X,U). Then the uniformly R-Lindelöfness of the mapping f implies there exists such open covering β of the space (Y,V) and uniformly locally finite open covering γ of the space (X,U), that the covering $f^{-1}\beta \wedge \gamma$ is refined in a covering α. Since $Y = \{y\}$, then $f^{-1}\beta \wedge \gamma = \gamma$. Let $\lambda \in U$ be an arbitrary uniform covering. Then exist such $\eta \in V$ and countable covering $\mu \in U$, that the covering $f^{-1}\eta \wedge \mu$ is refined in a covering λ. It's clear that $f^{-1}\eta \wedge \mu = \mu$. Hence, the space (X,U) is uniformly R-Lindelöf.

Lemma 1 *If α and β is uniformly locally finite covering of the space (X,U), then covering $\alpha \wedge \beta$ is uniformly locally finite covering of the space (X,U).*

Proof. Let α and β be a uniformly locally finite covering of the space (X,U).

We show the covering $\alpha \wedge \beta$ is also uniformly locally finite covering of the space (X,U). Since the coverings α and β is uniformly locally finite, there exists such uniform coverings $\mu \in U$ and $\eta \in U$, that $M \subset \bigcup_{i=1}^{n} A_i$, $N \subset \bigcup_{j=1}^{m} B_j$, $M \in \mu$, $N \in \eta$. Note that $M \bigcap N \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \bigcap B_j)$, $M \bigcap N \in \mu \wedge \eta$. Obviously, $\mu \wedge \eta$ is uniformly covering. Thus, the covering $\alpha \wedge \beta$ is uniformly locally finite.

Lemma 2 *Let $f : (X,U) \to (Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) onto a uniform space (Y,V). If β is uniformly locally finite open covering of the space (Y,V), then $f^{-1}\beta$ is uniformly locally finite open covering of the space (X,U).*

Proof. Let f be a uniformly continuous mapping of a uniform space (X,U) onto a uniform space (Y,V) and β be a uniformly locally finite open covering of the space (Y,V). Then exist such uniform covering $\alpha \in V$, that $|St(A,\beta)|$ is finite for all $B \in \beta$ i.e. for any $A \in \alpha$ exist such elements $B_i \in \beta$, $i = 1,2,...,n$, that $A \subset \bigcup_{i=1}^{n} B_i$. Consequently,$f^{-1}A \subset \bigcup_{i=1}^{n} f^{-1}B_i$, $f^{-1}A \in f^{-1}\alpha$, $f^{-1}B_i \in f^{-1}\beta$. Note, $f^{-1}\alpha \in U$ and $f^{-1}\beta$ is open covering of the uniform space (X,U). Thus, the covering $f^{-1}\beta$ is uniformly locally finite open covering of the space (X,U).

Theorem 4 *If f and (Y,V) is uniformly R-Lindelöf, then the uniform space (X,U) is uniformly R-Lindelöf.*

Proof. Let f and (Y,V) be a uniformly R-Lindelöf and α be an arbitrary open covering of the space (X,U). Then exist such open covering β of the space (Y,V) and uniformly locally finite open covering γ of the space (X,U), that the covering $f^{-1}\beta \wedge \gamma$ is refined in a covering α. By virtue of the uniformly R-Lindelöfness of the uniform space (Y,V) exist such uniformly locally finite open covering β_0, that the covering β_0 is refined in a covering β. Then $f^{-1}\beta_0 \wedge \gamma \succ f^{-1}\beta \wedge \gamma$. By virtue of Lemma 2 the open covering $f^{-1}\beta_0$ is uniformly locally finite. Denote $f^{-1}\beta_0 \wedge \gamma = \delta$. By virtue of Lemma 1 the open covering δ is uniformly locally finite. Let $\lambda \in U$ be an arbitrary uniform covering. Then exist such $\eta \in V$ and countable covering $\mu \in U$, that the covering $f^{-1}\eta \wedge \mu$ is refined in a covering λ. By virtue of the uniformly R-Lindelöfness of the uniform space (Y,V) exist such countable covering $\eta_0 \in V$, that the covering η_0 is refined in a covering η. Therefore $f^{-1}\eta_0 \wedge \mu \succ f^{-1}\eta \wedge \mu$. Denote $f^{-1}\eta_0 \wedge \mu = \omega$. Obviously, the covering $f^{-1}\eta_0$ and ω is countable. Hence, the space (X,U) is uniformly R-Lindelöf.

Definition 3 *A uniform space (X,U) is said to be uniformly Lindelöf, if for each finitely additive open covering γ of (X,U) there exists such a sequence of countable uniform coverings $\{\alpha_i : i \in N\} \subset U$, with property (*).*

Proposition 10 *Any uniformly R-Lindelöf space is uniformly Lindelöf.*

Proposition 11 *Any separable metrizable uniform apace (X,U) is uniformly Lindelöf.*

Proposition 12 *If (X,U) is uniformly Lindelöf space then the topological space (X,τ_U) is Lindelöf. Conversely, if (X,τ) is Lindelöf then the uniform space (X,U_X) is uniformly Lindelöf, where U_X is a universally uniformities of the space (X,τ).*

Proposition 13 *Each closed subspace of a uniformly Lindelöf space (X,U) is uniformly Lindelöf.*

Proposition 14 *Any uniformly Lindelöf space is uniformly locally Lindelöf.*

Proposition 15 *Any uniformly P -Lindelöf space is uniformly Lindelöf.*

Proposition 16 *Any compact uniform space is uniformly Lindelöf.*

Theorem 5 *Let (X,U) be a uniform space and bX be a certain compact Hausdorff extension of the space (X,τ_U). Then the following conditions are equivalent:*

1. *A uniform space (X,U) is uniformly Lindelöf.*
2. *For each compact $K \subset bX \backslash X$ there exist a sequence of countable uniformly coverings $\{\alpha_i\} \subset U$, realizing the condition: for each point $x \in X$ there exists such number $n \in N$, that $[\alpha_i(x)]_{bX} \bigcap K = \varnothing$.*

Proof. $1 \Rightarrow 2$. Let (X,U) be a uniformly Lindelöf space, bX be a certain compact Hausdorff extension of the space (X,τ_U) and $K \subset bX \backslash X$ be an arbitrary compact. Denote as λ the set of all such open subsets L of compact bX, that $[L]_{bX} \bigcap K = \varnothing$. Then it is easy to check that $\mu = \{L \bigcap X : L \in \lambda\}$ is finitely additive open covering of the space (X,U). By the condition 1, for the covering μ exists sequence of countable uniformly coverings $\{\alpha_i\}$, such that for each point $x \in X$ exist such number $i \in N$, that $\alpha_i(x) \subset L \bigcap X$ for each $L \in \lambda$. Consequently, $[\alpha_i(x)]_{bX} \bigcap K = \varnothing$.

$2 \Rightarrow 1$. Let μ be an arbitrary finitely additive open covering of the space (X,U). There exists such family λ of open subsets of the bX, that $\mu = \{L \bigcap X : L \in \lambda\}$. Denote $K = bX \backslash \bigcup\{L : L \in \lambda\}$. Consequently, for the compact $K \subset bX \backslash X$ there exist such sequence of countable uniformly coverings $\{\alpha_i\}$, that for each point $x \in X$ there exists such number $i \in N$, that $[\alpha_i(x)]_{bX} \bigcap K = \varnothing$. Then there exist such $\{L_1, L_2, ..., L_k\} \subset \lambda$, that $[\alpha_i(x)]_{bX} \subset \bigcup_{j=1}^{k} L_j$. Hence, $\alpha_i(x) \subset (\bigcup_{j=1}^{k} L_j) \bigcap X$. By virtue of finitely additiveness of coverings μ we have $(\bigcup_{j=1}^{k} L_j) \bigcap X \in \mu$. Thus, the uniform space (X,U) is uniformly Lindelöf.

Theorem 6 *The uniform space (X,U) is uniformly Lindelöf if and only if for each finitely additive open covering ω of the space (X,U) there exists a uniformly continuous ω -mapping f of the uniform space (X,U) onto a separable metrizable uniform space (Y,V) .*

Proof. *Necessity*. Let (X,U) be a uniformly Lindelöf space and ω be a finitely additive open covering of the space (X,U). Then for a covering ω there exists a normal sequence of the countable coverings $\{\alpha_i\} \subset U$, realizing the property (*). For sequence $\{\alpha_i\} \subset U$, there exists separable pseudometric d on X, such that $\alpha_{i+1}(x) \subset \{y : d(x,y) < \frac{1}{2^{i+1}}\} \subset \alpha_i(x)$, for all $x \in X$, $i \in N$. Introduce the relation of equivalence: $x \sim y$ if $d(x,y) = 0$, for any $x, y \in X$. Let Y_ω be the factor set of the set X relative to the equivalence relation " $\sim$ " and $f : X \to Y_\omega$ is natural projection. Denote $\rho(y_1,y_2) = d(f^{-1}y_1, f^{-1}y_2)$ for all $y_1, y_2 \in Y$. It is easy to check that ρ is a separable metric. Let V_ω be a uniformity on Y_ω induced by the separable metric ρ. The mapping $f : (X,U) \to (Y_\omega, V_\omega)$ is a uniformly continuous.

Let $y \in Y$ be an arbitrary point and $x \in f^{-1}y$. Then there exist such number $i \in N$ and $L \in \omega$, that $\alpha_i(x) \subset L$. Denote $O_y = \{y^* \in Y : \rho(y, y^*) < \frac{1}{2^{i+1}}\}$. Then $f^{-1}O_y \subset \{x \in X : \rho(x, y) \leq \frac{1}{2^{i+1}}\} \subset \alpha_i(x) \subset L$. Hence, f is a ω-mapping.

Sufficiency. Let ω be an arbitrary finitely additive open covering of the space (X,U) and $f : (X,U) \to (Y_\omega, V_\omega)$ be a uniformly continuous ω -mapping of the uniform space (X,U) onto a separable metrizable uniform space (Y_ω, V_ω). Then exists a base consisting of countable coverings $\{\gamma_i\} \subset V_\omega$. Then $\{f^{-1}\gamma_i\} \subset U$. It is easy to check that the sequence countable uniformly coverings $\{f^{-1}\gamma_i\}$ realizes the condition (*). Thus, the space (X,U) is uniformly Lindelöf.

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