On strongly uniformly paracompact spaces and mappings

Cite as: AIP Conference Proceedings **2325**, 020030 (2021); https://doi.org/10.1063/5.0040268
Published Online: 09 February 2021

Meerim Zhanakunova, and Bekbolot Kanetov

On Strongly Uniformly Paracompact Spaces and Mappings

Meerim Zhanakunova[1, a)] and Bekbolot Kanetov[2, b)]

[1)]Kuseyin Karasaev Bishkek Humanities University, Bishkek, Kyrgyz Republic

[2)]Jusup Balasagyn National University, Bishkek, Kyrgyz Republic

[a)]Electronic mail: aelin_jasmin@mail.ru

[b)]Corresponding author: bekbolot_kanetov@mail.ru

Abstract. In this article we study strongly uniformly *R*-paracompact space and strongly uniformly *R*-paracompact mappings. In particular, the characterizations of strongly uniformly *R*-paracompact spaces by using Hausdorff compact extensions and finite additive coverings are obtained.

INTRODUCTION

Throughout this work all uniform spaces are assumed to be Hausdorff, mappings are uniformly continuous.

For coverings α and β of the set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, for coverings α and β of a set X, we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$. The covering α finitely additive if $\alpha^{\angle} = \alpha$, $\alpha^{\angle} = \{\bigcup \alpha_0 : \alpha_0 \subset \alpha \text{ is finite}\}$. $\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(H) = \bigcup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha : A \bigcap H \neq \varnothing\}$, $H \subset X$.

A covering α of the uniform space (X,U) is called uniformly star finite if there exists a uniform covering $\beta \in U$ such that every $\alpha(B)$ meets α only for a finite number of elements of α; a covering α of the uniform space (X,U) is called uniformly locally finite if there exists a uniform covering $\beta \in U$ such that every $B \in \beta$ meets α only for a finite number of elements of α; the covering consisting of the union of countable number of (uniformly) locally finite families is called (uniformly) σ-locally finite; **s**tar finitely uniformly σ-locally finite coverings will be called uniformly σ-star finite; a uniform space (X,U) called uniformly *R*-paracompact, if every open covering has an open uniformly locally finite refinement [1]; a uniform space (X,U) called uniformly *B*-paracompact, if for each finitely additive open covering γ of (X,U) there exists such sequence uniform covering $\{\alpha_i : i \in N\} \subset U$, that following condition is realized: for each point $x \in X$ there exist such number $i \in N$ and $\Gamma \in \gamma$ that $\alpha_i(x) \subset \Gamma$ (*) [2]; a uniform space (X,U) called uniformly *P*-paracompact, if for each open cover γ of (X,U) there exists such sequence uniform covering $\{\alpha_i : i \in N\} \subset U$, that the condition (*) is realized [3]; a uniform space (X,U) is called strongly uniformly *R*-paracompact if every open covering has an open uniformly star finite refinement [2]; a uniform space (X,U) is called strongly uniformly *B*-paracompact if the space (X,U) is uniformly *B*-paracompact and (X,U) is strongly paracompact [5]; a uniform space (X,U) is called strongly uniformly *P*-paracompact if the space (X,U) is uniformly *P*-paracompact and (X,U) is strongly paracompact [5]; a uniform space (X,U) is called strongly uniformly paracompact, if every open cover of (X,U) has a uniformly σ-star finite open refinement [6].

A uniform space (X,U) is called uniformly *B*-Lindelöf, if it is both uniformly *B*-paracompact and $\aleph_0$-bounded [2]; a uniform space (X,U) is called $\aleph_0$-bounded if the uniformity U has a base consisting of countable coverings; a uniform space (X,U) is called uniformly *A*-Lindelöf, if for each open covering α exist a countable uniformly covering $\beta = \{B_n : n \in N\}$ and $\gamma \in U$ such that $\beta \succ \alpha^{\angle}$ and $\gamma(\bar{B}_n) \subset B_{n+1}$ for any $n \in N$ [7]; a uniform space (X,U) is called uniformly *P*- Lindelöf, if for each open cover γ of (X,U) there exists such sequence countable uniform covering $\{\alpha_i : i \in N\} \subset U$, that the condition (*) is realized [5]; a uniform space (X,U) is called uniformly *R*-Lindelöf, if it is both uniformly *R*-paracompact and $\aleph_0$-bounded; a uniformly continuous mapping $f : (X,U) \to (Y,V)$ of uniform space (X,U) onto a uniform space (Y,V) is called a precompact, if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and finite uniform covering $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$[2]; a uniformly continuous mapping $f : (X,U) \to (Y,V)$ of uniform space (X,U) onto a uniform space (Y,V) is called a uniformly perfect, if it is both precompact and perfect [2]; a uniformly continuous mapping $f : (X,U) \to (Y,V)$ of uniform space (X,U) onto a uniform space (Y,V) is called a uniformly open, if f maps each open uniform covering $\alpha \in U$ to an open uniform covering $f\alpha \in V$ [2]. For the uniformity U by τ_U we denote the topology generated by the uniformity and symbol U_X means the universal uniformity.

International Conference on Analysis and Applied Mathematics (ICAAM 2020)
AIP Conf. Proc. 2325, 020030-1–020030-5; https://doi.org/10.1063/5.0040268
Published by AIP Publishing. 978-0-7354-4069-2/$30.00

ON STRONGLY UNIFORMLY *R*-PARACOMPACT SPACES AND STRONGLY UNIFORMLY *R*-PARACOMPACT MAPPINGS

Let (X,U) be a uniform space.

Proposition 1 *If (X,U) is a strongly uniformly R-paracompact space then the topological space (X,τ_U) is strongly uniformly R-paracompact. Conversely, if (X,τ) is strongly paracompact then the uniform space (X,U_X) is strongly uniformly R-paracompact.*

Proof. Let α be an arbitrary open cover of the space (X,τ_U) . Then for the open covering α exists a uniformly star finite open covering β which is a refinement of it. Let $B\in\beta$ be an arbitrary set. Then exist $L\in\lambda$ such that $B\bigcap L\neq\varnothing$. Then $B\subset\beta(L)$ and the set $\beta(L)$ meets β only for a finite number of elements of β. Hence, β is star finite. Since the space (X,τ_U) is strongly uniformly *R*-paracompact. Consequently, the space (X,τ_U) is strongly paracompact.

Conversely, if (X,τ) be a strongly uniformly *R*-paracompact. Then the set of all open coverings forms a base of universal uniformity U_X of the space (X,τ). Thus, (X,U_X) is strongly uniformly *R*-paracompact.

Theorem 1 *A uniform space (X,U) is strongly uniformly R-paracompact if and only if every finitely additive open covering has an open uniformly star finite refinement.*

Proof. *Necessity*. Let (X,U) be a strongly uniformly *R*-paracompact space and α be an arbitrary finitely additive open covering of the space (X,U). Then exist uniformly star finite open covering β such that $\beta\succ\alpha$.

Sufficiency. Let α be an arbitrary open covering of the space (X,U). Then open covering $\alpha^{\angle}$ is finitely additive. Let β be an open uniformly star finite covering of the space (X,U) such that $\beta\succ\alpha^{\angle}$. For each $B\in\beta$ choose such $A^{\angle}\in\alpha^{\angle}$that $B\subset A^{\angle}$, where $A^{\angle}=\bigcup\limits_{i=1}^{n}A_i$, $A_i\in\alpha$, $i=1,2,...,n$. Denote $\beta_B=\{A_i\bigcap B: i=1,2,...,n\}$. Then β_B is an open uniformly star finite covering refined in the open covering α. Thus, (X,U) is strongly uniformly *R*-paracompact.

Theorem 2 *Any closed subspace uniform space (M,U_M) of a strongly uniformly R-paracompact space (X,U) is strongly uniformly R-paracompact.*

Proof. Let α_M be an arbitrary open covering of the subspace (M,U_M). Then there exits open family α of (X,U), such that $\alpha\wedge\{M\}=\alpha_M$. Denote $\beta=\{\alpha,X\backslash M\}$. Obviously the covering β is an open covering of the space (X,U). Then exist uniformly star finite open covering γ such that $\gamma\succ\beta$. Hence $\gamma_M\succ\alpha_M$. Then it is easy to check that γ_M is a uniformly star finite open covering of the space (M,U_M). Thus, (M,U_M) is strongly uniformly *R*-paracompact.

Theorem 3 *Let (X,U) be a uniform space and bX be a certain compact Hausdorff extension of the space (X,τ_U). A uniform space (X,U) is strongly uniformly R-paracompact if and only if for each compact $K\subset bX\backslash X$ there exist a uniformly star finite coverings α, such that $[A]_{bX}\bigcap K=\varnothing$ for any $A\in\alpha$.*

Proof. *Necessity*. Let (X,U) be a strongly uniformly *R*-paracompact space and $K\subset bX\backslash X$ be an arbitrary compact. Then for each point $x\in X$ there is a open neighborhood O_x of the bX, such that $[O_x]_{bX}\bigcap K=\varnothing$. Denote $\lambda=\{O_x\bigcap X: x\in X\}$. Obviously λ is open covering of the space (X,U). Let β be a uniformly star finite open covering of the space (X,U), refined in the covering λ, i.e. for any $B\in\beta$ there exists $O_x\bigcap X\in\lambda$ such that $B\subset O_x\bigcap X$. Then $[B]_{bX}\subset[O_x\bigcap X]_{bX}=[O_x]_{bX}$. Consequently, $[B]_{bX}\bigcap K=\varnothing$.

Sufficiency. Let α be an arbitrary finitely additive open covering of the space (X,U). Then for every $A\in\alpha$ exists open subsets $A\in\alpha$ of the bX such that $L_A\bigcap X=A$. Denote $K=bX\backslash\bigcup\{L_A: A\in\alpha\}$. Accordingly to the condition of the theorem, the exists a uniformly star finite coverings β, such that $[B]_{bX}\bigcap K=\varnothing$ for any $B\in\beta$. Since $[B]_{bX}$ is compact, there exist $L_{A_1},L_{A_2},...,L_{A_n}$ of the bX such that $[B]_{bX}\subset\bigcup\limits_{i=1}^{n}L_{A_i}$. Then $B\subset\bigcup\limits_{i=1}^{n}A_i$, $A_i\in\alpha$, $i=1,2,...,n$. Hence, $\alpha_i(x)\subset(\bigcup\limits_{j=1}^{k}L_j)\bigcap X$. By virtue of finitely additiveness of the coverings μ we have $(\bigcup\limits_{j=1}^{k}L_j)\bigcap X\in\mu$. Thus, the uniform space (X,U) is strongly uniformly *R*-paracompact.

Theorem 4 *A uniform space (X,U) is strongly uniformly R-paracompact if and only if uniform space (X,U) is uniformly R-paracompact and topological space (X,τ_U) is strongly paracompact.*

Proof. *Necessity*. The necessity is obvious.

Sufficiency. Let α be an arbitrary open covering of the space (X,U). By virtue of strongly paracompactness of uniform space (X,τ_U) there exist a open star finite coverings β, such that $\beta \succ \alpha$. Then exist an open uniformly locally finite covering γ refined in the covering β. The covering γ is uniformly locally finite, therefore exists a uniform covering $\lambda \in U$ such that every $L \in \lambda$ meets γ only for a finite number of elements of γ, i.e. the cardinality of the family $St(L,\gamma)$ is finite for each $L \in \lambda$. Let $L \in \lambda$ and $\Gamma \in St(L,\gamma)$. Since $\gamma \succ \beta$, then for any $\Gamma \in \gamma$ there exists $B \in \beta$ such that $\Gamma \subset B$. Due to the star finiteness of the covering β, it follows that the cardinality of the family $St(B,\beta)$ is finite for each $B \in \beta$ and even more so the cardinality of the family $St(\Gamma,\beta)$ is finite for each $\Gamma \in St(L,\gamma)$. Then the cardinality of the family $St(\gamma(L),\beta)$ is finite. Hence, the cardinality of the family $St(L,\beta)$ is finite for each $L \in \lambda$. Then it is easy to check that $St(\beta(L),\beta)$ is finite. Since, every $\beta(L)$ meets β only for a finite number of elements of β. Thus, the uniform space (X,U) is strongly uniformly R-paracompact.

Corollary 1 *Any compact uniform space (X,U) is strongly uniformly R-paracompact.*

Corollary 2 *Any uniformly locally compact uniform space (X,U) is strongly uniformly R-paracompact.*

Corollary 3 *Any uniformly R-paracompact and uniformly B-Lindelöf space (X,U) is strongly uniformly R-paracompact.*

Corollary 4 *Any uniformly R-Lindelöf space (X,U) is strongly uniformly R-paracompact.*

Corollary 5 *Any uniformly R-paracompact and uniformly A-Lindelöf space (X,U) is strongly uniformly R-paracompact.*

Corollary 6 *Any uniformly R-paracompact and uniformly P-Lindelöf space (X,U) is strongly uniformly R-paracompact.*

Corollary 7 *Any uniformly R-paracompact and strongly uniformly P-paracompact space (X,U) is strongly uniformly R-paracompact.*

Corollary 8 *Any uniformly R-Lindelöf space is strongly uniformly B-paracompact.*

Theorem 5 *Any strongly uniformly R-paracompact space is strongly uniformly B-paracompact.*

Proof. Let (X,U) be a strongly uniformly R-paracompact space. Then (X,U) is uniformly B-paracompact and its topological space (X,τ_U) is strongly paracompact. Thus, (X,U) is strongly uniformly B-paracompact.

Theorem 6 *Any strongly uniformly R-paracompact space is strongly uniformly paracompact.*

Proof. Let (X,U) be a strongly uniformly R-paracompact space. Then (X,U) is uniformly paracompact and its topological space (X,τ_U) is strongly paracompact. Thus, (X,U) is strongly uniformly paracompact.

Lemma 7 *A covering α of the uniform space (X,U) uniformly star finite if and only if it is uniformly locally finite and star finite.*

Proof. A uniformly locally finite follows directly from the definition of a uniformly star finite. Now let $A \in \alpha$ be an arbitrary element. Then exist $B \in \beta$ such that $A \bigcap B \neq \varnothing$. Then $A \subset \alpha(B)$ and the set $\alpha(B)$ meets α only for a finite number of elements of α. Hence, α is star finite. Conversely, let α be a uniformly locally finite and star finite covering. Then exists a uniform covering $\beta \in U$ such that every $B \in \beta$ meets α only for a finite number of elements of α i.e. exists $A_{i(B)} \in \alpha$ such that $B \subset \bigcup_{i=1}^{n} A_{i(B)}$. By virtue of the star finite of the covering α , every $A_{i(B)}$ meets α only for a finite number of elements of α. Then $\alpha(B)$ also meets α only for a finite number of elements of α. Thus, the covering α of the uniform space (X,U) uniformly star finite.

Lemma 1 imply the following.

Theorem 8 *A uniform space (X,U) is strongly uniformly R-paracompact if and only if every open covering has an open uniformly locally finite and star finite refinement.*

Lemma 1 and Theorem 2.3.9 [2, p. 155] imply the following theorem.

Theorem 9 *Let $f:(X,U)\to(Y,V)$ be a perfect mapping of a uniform space (X,U) onto a uniform space (Y,V). If (Y,V) is strongly uniformly R-paracompact space then the uniform space (X,U) is also strongly uniformly R-paracompact.*

Corollary 9 *Let $f : (X,U) \to (Y,V)$ be a uniformly perfect mapping of a uniform space (X,U) onto a uniform space (Y,V). If (Y,V) is strongly uniformly R-paracompact space then the uniform space (X,U) is also strongly uniformly R-paracompact.*

Proposition 2.2.15 [2, p. 145] and Exercise 5.3. H (d) [8, p. 487] imply the following theorem.

Theorem 10 *Let $f : (X,U) \to (Y,V)$ be a perfect uniformly open mapping of a uniform space (X,U) onto a uniform space (Y,V). If (X,U) is strongly uniformly R-paracompact space then the uniform space (Y,V) is also strongly uniformly R-paracompact.*

Definition 1 *A uniformly continuous mapping $f : (X,U) \to (Y,V)$ of a uniform space (X,U) onto a uniform space (Y,V) is said to be strongly uniformly R-paracompact, if for each open covering α of the space (X,U) there exist such open covering β of the space (Y,V) and uniformly star finite open covering α of the space (X,U), that the covering $f^{-1}\beta \wedge \alpha$ is refined in a covering λ;*

Proposition 2 *Let $f : (X,U) \to (Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) onto a uniform space (Y,V). If (X,U) is strongly uniformly R-paracompact space then the uniformly continuous mapping f is strongly uniformly R-paracompact.*

Proof. Let (X,U) be a strongly uniformly R-paracompact space and λ be an arbitrary open covering. Then exist such uniformly star finite open covering γ of the space (X,U), that the covering α is refined in a open covering λ. For open covering β of the space (Y,V) we have the covering $f^{-1}\beta \wedge \alpha$ is refined in a covering λ. Consequently, the mapping f is strongly uniformly R-paracompact.

Proposition 3 *If uniformly continuous mapping $f : (X,U) \to (Y,V)$ of a uniform space (X,U) onto a uniform space (Y,V) , $Y = \{y\}$ is strongly uniformly R-paracompact, then the uniform space (X,U) is strongly uniformly R-paracompact.*

Proof. Let f be a strongly uniformly R-paracompact mappings and λ be an arbitrary open covering of the space (X,U). Then exists such open covering β of the space (Y,V) and uniformly star finite open covering α of the space (X,U), that the covering $f^{-1}\beta \wedge \alpha$ is refined in a covering λ. Since $Y = \{y\}$, then $f^{-1}\beta \wedge \alpha = \alpha$. Thus, (X,U) is strongly uniformly R-paracompact.

Lemma 11 *If α and β is uniformly star finite covering of the space (X,U), then covering $\alpha \wedge \beta$ is uniformly star finite covering of the space (X,U).*

Proof. Let α and β be a uniformly star finite covering of the space (X,U). We show the covering $\alpha \wedge \beta$ is also uniformly star finite covering of the space (X,U). Since the coverings α and β is uniformly star finite, there exists such uniform coverings $\mu \in U$ and $\eta \in U$, that $\alpha(M) \subset \bigcup\limits_{i=1}^{n} A_i$, $\beta(N) \subset \bigcup\limits_{j=1}^{m} B_j$, $M \in \mu$, $N \in \eta$. Note that $(\alpha \wedge \beta)(M \bigcap N) \subset \alpha(M) \bigcap \beta(N) \subset \bigcup\limits_{i=1}^{n} \bigcup\limits_{j=1}^{m} (A_i \bigcap B_j)$, $M \bigcap N \in \mu \wedge \eta$. Obviously, $\mu \wedge \eta$ is uniformly covering. Thus, the covering $\alpha \wedge \beta$ is uniformly star finite.

Lemma 12 *Let $f : (X,U) \to (Y,V)$ be a uniformly continuous mapping of a uniform space (X,U) onto a uniform space (Y,V). If β is uniformly star finite open covering of the space (Y,V), then $f^{-1}\beta$ is uniformly star finite open covering of the space (X,U).*

Proof. Let f be a uniformly continuous mapping and β be a uniformly star finite open covering of the space (Y,V). Then exist such uniform covering $\alpha \in V$, that $|St(\alpha(B),\beta)|$ is finite for all $B \in \beta$ i.e. for any $B \in \beta$ exist such elements $B_i \in \beta$, $i = 1,2,...,n$, that $\alpha(B) \subset \bigcup\limits_{i=1}^{n} B_i$. Since f is a uniformly continuous mapping, then $f^{-1}\beta$ is open covering of the space $(X,U)$and $f^{-1}\alpha \in U$. Consequently, $f^{-1}\alpha(B) \subset \bigcup\limits_{i=1}^{n} f^{-1}B_i$, $f^{-1}B_i \in f^{-1}\beta$. Thus, the covering $f^{-1}\beta$ is uniformly star finite open covering of the space (X,U).

Theorem 13 *If f and (Y,V) is strongly uniformly R-paracompact, then the uniform space(X,U) is strongly uniformly R-paracompact.*

Proof. Let f and (Y,V) be a strongly uniformly R-paracompact and λ be an arbitrary open covering of the space (X,U). Then exist such open covering β of the space (Y,V) and uniformly star finite open covering α of the space (X,U), that the covering $f^{-1}\beta \wedge \alpha$ is refined in a covering λ. By virtue of the strongly uniformly R-paracompactness of the uniform space (Y,V) exist such uniformly star finite open covering β_0, that the covering β_0 is refined in a covering β. Obviously, $f^{-1}\beta_0 \wedge \alpha \succ f^{-1}\beta \wedge \alpha$. By virtue of Lemma 3 the open covering $f^{-1}\beta_0$ is uniformly star finite. Denote $f^{-1}\beta_0 \wedge \gamma = \delta$. By virtue of Lemma 2 the open covering δ is uniformly star finite. Hence, the space (X,U) is strongly uniformly R-paracompact.

REFERENCES

1. M. D. Rice, Proc. Amer. Math. Soc **62:2**, 359-362 (1977).
2. A. A. Borubaev, *Uniform Topology* (Ilim, Frunze, 2013) (in Russian).
3. D. Buhagiar and B. Pasynkov, Czech Math. J. **46(121)**, 577-586 (1996).
4. B. E. Kanetov, *Some Classes of Uniform Spaces and Uniformly Continuous Mappings* (Bishkek, 2013) (in Russian).
5. B. E. Kanetov and N. A. Baigazieva, AIP Conference Proc. **1997**, 020023 (2018).
6. B. E. Kanetov, D. E. Kanetova, and N. A. Baigazieva, On one property of uniform space, in: Internat. Conf. Topology Appl., Nafpactos, Greece, 2018, pp. 60.
7. L. V. Aparina, Trudy Mosk. Math. Obshestva **57**, 3-15 (1996) (in Russian).
8. R. Engelking, *General Topology* (Mir, Moscow, 1986) (in Russian).